Kawaguchi-Silverman conjecture for int-amplified endomorphisms

(Joint work with Sheng Meng)

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Contents

1. Background & Terminology (Dynamical degrees)

- Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X of dimension n over $\overline{\mathbb{Q}}$, and H an ample divisor on X.
- There are two fundamental invariants for f: Dynamical degrees and Arithmetic degrees.
- The p-th dynamical degree $d_p(f)$ $(0 \leqslant p \leqslant n)$ of f is defined as

$$d_p(f) := \lim_{m\to\infty}((f^m)^* H^p \cdot H^{n-p})^{1/m}.$$

  Such a limit exists and is independent of the choice of H (Dinh-Sibony, 05), (Dang, 20), (Truong, 20).
- For smooth $X/\mathbb{C}$, $d_p(f) =$ the spectral raidus of $f^*|_{H^{p,p}(X,\mathbb{C})}$.
- For $p = 1$, $d_1(f) =$ the spectral raidus of $f^*|_{\mathrm{N}^1(X)}$.
- Dynamical degrees satisfy log concavity: $\exists 1 \leqslant u \leqslant v \leqslant n$ s.t. $1 = d_0(f) < \cdots < d_u(f) = \cdots = d_v(f) > \cdots > d_n(f) = \deg(f)$.
- The last dynamical degree $d_n(f)$ is the topological degree of f.
- f has dominant topological degree if $\forall k \leqslant n-1$, $d_n(f) > d_k(f)$.

1. Background & Terminology (Dynamical degrees)

Definition (Int-amplified endomorphisms) (Meng, 20), (Matsuzawa)

A surjective endomorphism $f: X \to X$ of a projective variety X is called int-amplified, if all the e.vs of the linear operation $f^*|_{\mathrm{N}^1(X)}$ are of modulus $> 1 \iff f^*H - H$ is ample for some ample divisor $H \iff f$ has a dominant topological degree.

Example

- Consider the power map $f: \mathbb{P}^n \to \mathbb{P}^n$:
$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots : x_n^q].$$
Then $f^*H \sim qH$ for any hyperplane H on X. It is straightforward to show $d_p(f) = q^p$ for each $p = 0, 1, \cdots, n$.
- Projective toric/abelian varieties admit int-amplified endomorphisms.
- The product of two int-amplified endomorphisms is int-amplified.

1. Background & Terminology (Arithmetic degrees)

Theorem (Weil's Height Machine)

There exists a unique homomorphism

$$h_X\colon \mathrm{Pic}(X)_{\mathbb{R}} \to \{\text{functions } X(\overline{\mathbb{Q}}) \to \mathbb{R}\}/\{\text{bounded functions } X(\overline{\mathbb{Q}}) \to \mathbb{R}\}$$

satisfying the following properties.

- *Let D be a very ample divisor on X and $\phi_D\colon X \hookrightarrow \mathbb{P}^N$ the associated embedding. Then we have*

  $$h_{X,D} = h \circ \phi_D + O(1),$$

  where h is the absolute logarithmic height on $\mathbb{P}^N$.
- *Let $\pi\colon X \to Y$ be a morphism of smooth projective varieties and $D_Y \in \mathrm{Pic}(Y)_{\mathbb{R}}$. Then we have*

  $$h_{X,\pi^*D_Y} = h_{Y,D_Y} \circ \pi + O(1).$$

1. Background & Terminology (Arithmetic degrees)

Theorem (Weil's Height Machine (continued))

- *Let D_1, D_2 be $\mathbb{R}$-divisors on X. Then we have*

$$h_{X,D_1+D_2} = h_{X,D_1} + h_{X,D_2} + O(1).$$

- *Let $D \geqslant 0$ be an integral divisor on X. Then $h_{X,D} \geqslant O(1)$ outside the base locus $\mathrm{Bs}(D)$ of D.*
- *Let $H, D \in \mathrm{Pic}(X)_{\mathbb{R}}$ be $\mathbb{R}$-divisors with H ample and D algebraically equivalent to zero. Then there is a constant $C > 0$ such that*

$$h_{X,D} \leqslant C\sqrt{h^+_{X,H}},$$

where $h^+_{X,H} := \max(1, h_{X,H})$.

The terms $O(1)$ only depend on varieties, divisors, and morphisms, but are independent of rational points of varieties.

1. Background & Terminology (Arithmetic degrees)

Definition (Arithmetic degrees)

Let $h^+_{X,H}$ be the modified Weil height as above. For each $x \in X(\overline{\mathbb{Q}})$, define the arithmetic degree of f at x:

$$\alpha_f(x) := \lim_{m\to\infty} h^+_{X,H}(f^m(x))^{1/m} \in \mathbb{R}_{\geqslant 1}.$$

From (Kawaguchi-Silverman, 16), the limit exists and is independent of the choice of the ample divisor H.

(KS16) shows $\alpha_f(x) \leqslant d_1(f)$: the arithmetic complexity of the forward orbit of x does not exceed the geometric complexity of iterations of f.

Conjecture (Kwaguchi-Silverman Conjecture=KSC)

Let $f : X \to X$ be a surjective endomorphism. Then for any $x \in X(\overline{\mathbb{Q}})$, if the forward f-orbit $\mathcal{O}_f(x) := \{f^m(x) : m \in \mathbb{Z}_{\geqslant 0}\}$ of x is Zariski dense in X, then $\alpha_f(x) = d_1(f)$.

2. Conjecture & Previous results

Conjecture (Kwaguchi-Silverman Conjecture=KSC)

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There are two main approaches to attack this conjecture. One is the well-developed technique: the construction of the canonical height (Call-Silverman, 91). In this direction, the conjecture is known in the following cases:

- f is polarized, i.e., $f^*H \sim qH$ for some ample divisor H and $q > 1$ (Kawaguchi-Silverman, 16);
- X is a smooth surface (KS, 14), (Matsuzawa-Sano-Shibata, 18);
- X is a Mori dream space (Matsuzawa, 20);
- X is an abelian variety (Kawaguchi-Silverman, 16), (Silverman, 17);
- X is a hyperKähler variety (Lesieutre-Satriano, 21);

2. Conjecture & Previous results

The other approach is based on the equivariant minimal model program (Meng-Zhang, 18, 20).

- $\dim(X) = 2$ (Meng-Zhang, 22);
- $\dim(X) = 3$, X smooth and $\deg(f) > 1$ (Meng-Zhang, 23), (LS, 21);
- X is smooth rationally connected admitting an int-amplified endomorphism (Matsuzawa-Yoshikawa, 22).

Theorem (Meng-Zhang, 22)

Let X be a $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism. Then we have:

(1) If K_X is psef, then KSC ($\checkmark$) for any surjective endomorphism of X.

(2) Suppose that KSC holds for the Case TIR. Then KSC ($\checkmark$) for any surjective endomorphism f of X.

RMK: As observed by J. Moraga, the $\mathbb{Q}$-factorialization of a projective klt variety is equivariant. Hence, one can remove the $\mathbb{Q}$-factorial assump. by some further analysis. Here, we focus on $\mathbb{Q}$-factorial varieties for simplicity.

2. Conjecture & Previous results

What is Case TIR?

The case TIR (Totally invariant ramification case):

Let X be a normal projective variety of dimension $n \geqslant 1$, which has only $\mathbb{Q}$-factorial klt singularities and admits an int-amplified endomorphism. Let $f \colon X \to X$ be an arbitrary surjective endomorphism. Moreover, we impose the following conditions.

(A1) The anti-Kodaira dimension $\kappa(X, -K_X) = 0$; $-K_X$ is nef, whose class is extremal in both the *nef cone* and the *pseudo-effective cone*.

(A2) $f^*D = d_1(f)D$ for some prime divisor $D \sim_{\mathbb{Q}} -K_X$.

(A3) The ramification divisor of f satisfies $\operatorname{Supp} R_f = D$.

(A4) There is an f-equivariant Fano contraction $\tau \colon X \to Y$ with $d_1(f) > d_1(f|_Y)$ $(\geqslant 1)$.

We aim to prove that the case TIR does not occur up to a finite cover so as to finish the int-amplified case for KSC. Primarily using A2 and A3!

2. Conjecture & Previous results

Why reduce to the case TIR?

Proposition (Meng-Zhang, 22)

Let X be a $\mathbb{Q}$-factorial projective klt variety admitting an int-amplified endomorphism. Let $f\colon X \to X$ be a surjective endomorphism. Then there exists an f-equivariant MMP: a composition of birational MMP $X \dashrightarrow X'$ followed by a Fano contraction $X' \to Y$ such that one of the following holds.

- $f^*K_X \equiv d_1(f)K_X$ with $d_1(f) > 1$ and $\kappa(X, -K_X) > 0$.
  (✓) f-equivariant anti-Kodaira fibration $X \dashrightarrow Z$ & $f|_Z$ is polarized.
- $\dim(Y) < \dim(X)$ and $d_1(f) = d_1(f|_Y)$.
  (✓) Induction.
- $d_1(f) > d_1(f|_Y)$; $\kappa(X, -K_X) = 0$ so $-K_X \sim_{\mathbb{Q}} D$; the class of $-K_X$ is extremal in both the nef cone and the pseudo-effective cone; and $D = \operatorname{Supp} R_f$ is a prime divisor with $f^*D = d_1(f)D$.
  (Remains open).

3. Main results (KSC)

Main Theorem (Meng-Z., 24)

Let X be a $\mathbb{Q}$-factorial klt projective variety admitting an int-amplified endomorphism. Then the Kawaguchi-Silverman conjecture holds for any surjective endomorphism f of X.

$(*)$ Let X be a normal projective variety and D a reduced divisor. Denote by $\mathrm{SEnd}(X, D)$ the monoid of surj. endo. f of X with $f^{-1}(D) = D$.

Theorem (Equivariant cover (MZ24); cf. (Moraga-Yáñez-Yeong, 24))

Assume $\mathrm{SEnd}(X, D)$ *has an int-amplified* $\mathcal{I}$ *such that* $\mathcal{I}|_{X \setminus D}$ *is quasi-étale. Then* $\forall f \in \mathrm{SEnd}(X, D)$, $\exists$ *a quasi-étale cover* $\pi : \widehat{X} \to X$ *satisfying:*

1. $(\widehat{X}, \pi^* D)$ *admits a splitting toric fibration over an abelian variety,*
2. $\exists \widetilde{f} \in \mathrm{SEnd}(\widehat{X}, \pi^* D)$ *such that* $\pi \circ \widetilde{f} = f^s \circ \pi$ *for some* $s > 0$, *and*
3. $\mathrm{SEnd}(\widehat{X}, \pi^* D)$ *contains an int-amplified endomorphism (but it may not be the lifting of* $\mathcal{I}$*!).*

3. Main results (totally invariant ramifications)

Definition: Splitting toric fibration

Let $\pi\colon (X, D) \to Y$ be a fibration between normal varieties where D is a Weil $\mathbb{Q}$-divisor on X. We call π is splitting toric fibration if (X, D) is (analytically) locally trivial over Y, for any fibre F of π, $(F, D|_F)$ is a toric pair, and for each irreducible component D_i of D, the restriction $D_i|_F$ is irreducible for a general fibre F of π.

Proposition (Meng-Z., 24); (Meng-Xie-Zhang, 25)

If π is a splitting toric fibration, then π is Zariski locally trivial.

- When X is a Fano manifold of Picard number 1, (Hwang-Nakayama, 11) proved that X is the projective space.
- When X is smooth rationally connected, (Meng-Zhang, 19) and (Meng-Z., 23) proved that X is a toric variety.
- The toric or toric fibration structure for the case TIR has also been proved by (Moraga-Yáñez-Yeong, 24) with a more general singularity condition and with a different approach.

3. Main results (Equivariant resolution)

Caution: the upstair of the equivariant toric cover is neither smooth nor $\mathbb{Q}$-factorial. In certain good case, we can take a higher smooth model:

Theorem (Equivariant resolution of toric covers) (Meng-Z., 24)

Let $\widetilde{X}$ be a normal projective variety and $\widetilde{D} \geqslant 0$ a reduced divisor. Let $\pi\colon (\widetilde{X}, \widetilde{D}) \to A$ be a Zariski locally trivial toric fibration over an abelian variety A. Then $\exists$ a resolution $\sigma\colon \widehat{X} \to \widetilde{X}$ s.t. $\forall \widetilde{f} \in \mathrm{SEnd}(\widetilde{X}, \widetilde{D})$ with $\widetilde{f}^*\widetilde{D} = q\widetilde{D}$, $\exists \widehat{f} \in \mathrm{SEnd}(\widehat{X}, \sigma^{-1}(\widetilde{D}))$ with $\sigma \circ \widehat{f} = \widetilde{f}^s \circ \sigma$ for some $s > 0$.

By the above theorems, for the case TIR, up to an equivariant generically finite cover $\widetilde{X} \to \widehat{X} \to X$, we may continue to run the equivariant MMP

$$\widetilde{X} = \widetilde{X}_1 \overset{\widetilde{\pi}_1}{\dashrightarrow} \widetilde{X}_2 \overset{\widetilde{\pi}_2}{\dashrightarrow} \cdots \overset{\widetilde{\pi}_{s-1}}{\dashrightarrow} \widetilde{X}_s \xrightarrow{\widetilde{\tau}} \widetilde{Y}$$

Here, each $\widetilde{\pi}_i$ is birational, and $\widetilde{\tau}$ is the first Fano contraction. Note that $\widetilde{X}_s \to A$ remains a splitting smooth toric fibration. Hence, $-K_{\widetilde{X}_s} \sim_{\mathbb{Q}}$ some reduced divisor with at least two components (toric boundary property). Then the irreducible boundary assumption of Case TIR is not satisfied!

4. Ideas of proofs

We shall only discuss the toric bundle structure. Lifting of an int-amplified endo. and a surjective endo. to the same splitting toric cover is technical, and we omit this part.

Set-up

Let $f\colon X \to X$ be an int-amplified endomorphism of a projective klt variety and D a reduced divisor such that $f^{-1}(D) = D$ and f is quasi-étale away from the support of D. The latter is equivalent to $K_X + D \equiv 0$.

Theorem (Druel-Lo Bianco, 2022); cf. (Meng-Z., 2024)

Let X be a projective klt variety over $k = \overline{k}$ of char $k = 0$. Let $D \geqslant 0$ be a reduced divisor s.t. $\Omega_X^{[1]}(\log D)$ is numerically flat locally free. Then the augmented irregularity $\tilde{q}(X) < \infty$ holds. Suppose further that (X, D) is lc with $q(X) = \tilde{q}(X)$. Then the Albanese morphism $(X, D) \to A$ is an analytically locally trivial toric fibration.

GOAL: After a quasi-étale cover, $\mathcal{T}_X(-\log D)$ is numerically flat.

4. Ideas of proofs

How to realize it?

Proposition (Iwai-Matsumura-Z., 23), (Meng-Z., 24)

Let X be a projective klt variety over $k = \overline{k}$ of char $k = 0$, and $\mathcal{E}$ a psef reflexive sheaf on X. If $\det(\mathcal{E}) \cdot H^{\dim(X)-1} = 0$ for an ample divisor H, then $\exists$ a quasi-étale cover s.t. the reflexive pullback of $\mathcal{E}$ is numerically flat.

- The above proposition extends (Cao-Höring, 19), (Höring-Peternell, 19), and (Hosono-Iwai-Matsumura, 22) to deduce the structures for klt varieties w/ certain positivity.
- The above psef is in the stronger sense and implies $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ psef.
- Once we apply the above theorem to show the numerical flatness of $\mathcal{T}_X(-\log D)$ after a quasi-étale cover, we can then conclude the structure essentially by (Druel-Lo Bianco, 2022).
- As $K_X + D \equiv 0$ by our assumption, to apply the above proposition to show the numerical flatness of $\mathcal{T}_X(-\log D)$ after a quasi-étale cover, we are left to prove the pseudo-effectiveness of $\mathcal{T}_X(-\log D)$.

4. Ideas of proofs

Indeed, without assuming $K_X + D \equiv 0$, a more general result holds true.

Theorem (Z., 25), (Meng-Z., 24)

Let $f\colon X \to X$ be an int-amplified endomorphism of a normal projective variety X and D a reduced divisor with $f^{-1}(D) = D$. Then the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ (and hence $\mathcal{T}_X$) is psef. Moreover, if X is smooth rationally connected, then $\mathcal{T}_X$ is generically ample.

Definition (Positivity of reflexive sheaves)

Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal projective variety X.

- $\mathcal{E}$ is pseudo-effective (psef) if $\forall a \in \mathbb{Z}_{>0}$ and $\forall$ ample divisor A, $\exists b \in \mathbb{Z}_{>0}$ such that $\mathrm{Sym}^{[ab]}\mathcal{E} \otimes \mathcal{O}_X(bA)$ is globally generated at some point (Viehweg, 83), (Nakayama, 04); $\Rightarrow \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ psef.
- $\mathcal{E}$ is nef if the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef.
- $\mathcal{E}$ is numerically flat if both $\mathcal{E}$ and its dual $\mathcal{E}^\vee$ are nef and locally free.

4. Ideas of proofs

We briefly explain the rough idea towards the proof (for simplicity, $D = 0$):

- Use the injection $0 \to \mathcal{T}_X \to (f^m)^{[*]}\mathcal{T}_X$, $\forall m \in \mathbb{N}$. Note that $f^*H - H$ is ample for some ample divisor H.
- We aim to show $\mathrm{Sym}^{[s]}\mathcal{T}_X \otimes \mathcal{O}_X(H)$ is psef for any s;
- Then "closedness" implies $\mathrm{Sym}^{[s]}\mathcal{T}_X$ itself is psef;
- Fix $\mathrm{Sym}^{[s]}\mathcal{T}_X$, choose a sufficiently ample $H^{\otimes n_s}$ s.t. $\mathrm{Sym}^{[s]}\mathcal{T}_X \otimes H^{\otimes n_s}$ is globally generated. Then for any $m \in \mathbb{N}$, the sheaf

$$((f^m)^*\mathrm{Sym}^{[s]}\mathcal{T}_X) \otimes H^{\otimes n_s}$$

  is generically globally generated;
- As $(f^m)^*H - H$ is sufficiently ample for sufficiently large m, if we put "$H^{\otimes n_s}$" into the bracket, then it would be "rather small ample";
- Suppose X is smooth rationally connected but $\mathcal{T}_X$ is not generically ample. Then $\exists$ a quotient bundle (still being psef) with trivial det. Hence, it is a direct sum of trivial line bundles and Ω_X contains a section, a contradiction.

5. Further discussions

The above theorem has some other applications:

Conjecture (Fakhruddin, 03)

Let X be a rationally smooth projective variety admitting an int-amplified endomorphism. Then X is toric.

- The conjecture is known for surfaces (Nakayama, 02); Fano threefolds (Meng-Zhang-Z., 22); (Totaro, 24); (Kawakami-Totaro, 25); $\dim(X) = 3$ and $\rho(X) = 2$ (Chen-Meng-Z.,25).
- Projective toric manifolds have big tangent bundles (Hsiao, 15). So we give the first step of this expectation: psef.
- (Höring-Liu-Shao, 22) proved that a smooth hypersurface of deg $\geqslant 3$ has non-psef tangent bundle. So we reprove:

Theorem (Beauville, 01) and (Paranjape-Srinivas,89)

Smooth hypersurf. deg $\geqslant 2$ and dim $\geqslant 3$ admit no non-isom. surj. endo.

5. Further discussions (LC singularities?)

Two further questions naturally appeared. The first question is to extend our theorem to log canonical singularities.

Question (Log canonical singularities)

Can we extend the result to the log canonical singularities? That is, if $f\colon X \to X$ is an int-amplified endomorphism of a projective $\mathbb{Q}$-Gorenstein variety, does KSC hold for (X, f)?

- By (Meng, 20), such a variety X has only lc singularities. However, even if we assume X is $\mathbb{Q}$-factorial, we do not know whether we can run the (equivariant) minimal model program.
- The construction of an f-equivariant dlt modification may help us reduce to the study of EMMP for dlt singularities (not known).
- Applying the equivariant anti-Kodaira fibration constructed by (Meng-Zhang, 22), it is possible to handle the case $\kappa(X, -K_X) > 0$.
- However, the characterization of the case TIR remains unclear.

5. Further discussions (ZDO ?)

The second question is on the assumption of KSC.

Conjecture (Medvedev-Scanlon, 09; Amerik-Bogomolov-Rovinsky, 11)

Let X be an irreducible variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $f : X \dashrightarrow X$ be a dominant rational self-map. If there exists no f-invariant rational functions, then there exists $x \in X(\mathbf{k})$ whose orbit is well-defined and Zariski dense in X.

- This conjecture is called the Zariski dense orbit conjecture (ZDO).
- ZDO is known for endomorphisms or birational automorphisms in dimension two due to (Xie, 25). Little is known in higher dimensions.
- Different from the KSC, the endomorphism of projective spaces is one of the most troublesome cases for ZDO.
- Under the assumption that f is int-amplified, by the dynamical degree calculations, there exists no f-invariant rational function, and hence ZDO implies that the assumption of KSC is satisfied.

Thank you for your attention!